【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (10%) Let V and W be vector spaces over the field of rational numbers. Let $T : V \to W$ be a function from V to W. Suppose $T(x+y) = T(x) + T(y)$ for all $x, y \in V$. Prove or disprove that T is a linear transformation.

2. Let $\mathfrak{M}_{3\times3}(\mathbb{R})$ be the set of all 3×3 matrices whose entries are real. Let $\delta : \mathfrak{M}_{3\times3}(\mathbb{R}) \to \mathbb{R}$ be a function satisfying the following.

   - δ is a linear function of each row when the remaining 2 rows are held fixed.
   - $\delta(A) = 0$ whenever two adjacent rows of A are identical.
   - $\delta(I_3) = 1$, where I_3 is the 3×3 identity matrix.

   (a) (5%) Let $A = I_3$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Prove that $\delta(A) = -\delta(B)$.

   (b) (5%) Let $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$. Find $\delta(C)$.

3. (10%) Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear. If $\mathsf{rank}(T) = \mathsf{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Here $R(T)$ and $N(T)$ are, respectively, the range and the null space of T.

4. (10%) Let $V$and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathfrak{L}(V, W)$. Here $\mathfrak{L}(V, W)$ is the set of all linear transformations from V to W.

5. (10%) Consider the linear system $Ax = b$, where $A \in \mathfrak{M}_{3\times5}(\mathbb{R})$, $x = (x_1\ x_2\ x_3\ x_4\ x_5)^T \in \mathbb{R}^5$ and $b = (b_1\ b_2\ b_3)^T \in \mathbb{R}^3$. Here v^T denotes the transpose of the vector v. Let the first, second and fourth columns of A be $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, respectively. Suppose $\left\{ \begin{pmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$ is a basis for the solution of the corresponding homogeneous system $Ax = 0$. Does there exist an A so that the above conditions are met? Is such A unique? Justify your answers.

6. Let A be a 3×3 matrix over the field of real numbers with eight of nine entries given as below:

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & ? \\ 3 & 9 & -3 \end{pmatrix}.$$

Let $\lambda \neq 0$ be a fixed real number and suppose that one of the eigenvalues of A is equal to λ.

(a) (5%) Find all the other eigenvalues of A and give logical reasons for your answer.

(b) (5%) List all values of λ for which A is not diagonalizable and give logical reasons for your answer.

7. Determine whether the following sets V with $\langle -, - \rangle$ are inner product spaces and give logical reasons for your answers.

(a) (5%) Let $V = C([0,1])$, the set of all continuous real-valued functions on the unit interval $[0,1]$, and for all $f, g \in V$, define $\langle f, g \rangle = \int_0^{\frac{1}{2}} f(x)g(x)dx$.

(b) (5%) Let $V = \mathbb{R}^2$ and for all $x, y \in V$, define $\langle x, y \rangle = x \cdot Ay$, where $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, and $v \cdot w$ denotes the dot product of v and w on $\mathbb{R}^2$.

(c) (5%) Let V be the set of all 2×2 matrices over the field of real numbers and for all $A, B \in V$, define $\langle A, B \rangle = \text{trace}(A + B)$, where trace$(C)$ denotes the trace of a matrix C.

(d) (5%) Let V be the set of all real-valued polynomials with real coefficients, and for $p, q \in V$, define $\langle p, q \rangle = \int_0^1 p'(t)q(t)dt$, where $r'(t)$ denotes the derivative of a function r at t.

8. Let V be a finite-dimensional inner product space with $\langle -, - \rangle$ over the field of complex numbers. Fix two vectors y, z in V, define a linear map $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for $x \in V$.

(a) (5%) Prove that T is linear and that there exists a unique linear map $T^* : V \to V$ such that $\langle u, T^*w \rangle = \langle Tu, w \rangle$ for all $u, w \in V$.

(b) (5%) Find an explicit expression for T^* and give logical reasons for your answer.

9. (10%) Let A be a 3×3 matrix over the field of complex numbers for which A^3 is the zero matrix, but A^2 is not the zero matrix. Find the Jordan canonical form of A and give logical reasons for your answer.