國立交通大學 103 學年度碩士班考試入學試題

科目：高等微積分(4031) 考試日期:103年2月14日 第 3 節

系所班別：應用數學系 組別：應數系甲組 第 1 頁,共 1 頁

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Decide which of the following statements are true and which are false. Prove the true ones and provide counterexamples for the false ones.
   (a) (6 points) Let F_k be a sequence of bounded nonempty sets in $\mathbb{R}^n$ such that $F_{k+1} \subseteq F_k$ for all $k = 1, 2, \ldots$. Then $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.
   (b) (6 points) Let $f : (a, b) \to \mathbb{R}$ be one-to-one. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f^{-1} is differentiable in $f((a, b))$.
   (c) (6 points) Let $a_k \in \mathbb{R}$ and suppose that $\limsup_{k\to\infty} |a_k|^{1/k} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
   (d) (6 points) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and that there is a $K > 0$ such that
   $$\|f(x) - f(y)\| \geq K\|x - y\|$$
   for all $x, y \in \mathbb{R}^n$. Then $\det(Df(x)) \neq 0$ for all $x \in \mathbb{R}^n$.

2. (a) (8 points) Show that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
   (b) (8 points) Suppose that f is integrable on $[0, 1]$. Show that
   $$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$

3. (12 points) Find $\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n^2} \frac{1}{\sqrt{n^2 + nj + k}}$. Justify your computation.

4. Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{x}{|y|\sqrt{x^2+y^2}} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
   (a) (14 points) Compute the directional derivatives of f at $(0, 0)$, where they exist. Justify your computation.
   (b) (12 points) Is f differentiable at $(0, 0)$? Justify your answer.

5. (10 points) Let f_n be integrable on $[0, 1]$ and $f_n \to f$ uniformly on $[0, 1]$. Show that if $\{b_n\}_{n=1}^{\infty}$ is increasing and converges to 1, then
   $$\lim_{n\to\infty} \int_0^{b_n} f_n(x)dx = \int_0^1 f(x)dx.$$

6. (12 points) Find the positive oriented simple closed smooth curve C on the plane $\mathbb{R}^2$ for which the *line integral* $\int_C (y^3 - y)\, dx - 2x^3\, dy$ is a maximum. Justify your assertion.