1. (a) Prove that the set of elements a_i in a group G such that $a_i x = x a_i$ for all x in G is a subgroup of G. (10 points)

(b) Find the inverse of the matrix $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. (5 points)

2. (a) Show that the Fourier transform of a convolution $\int_{-\infty}^{+\infty} f_1(x) \cdot f_2(y - x)\, dx$

is equal to $F_1(k) \cdot F_2(k)$, where $F_i(k) = \int_{-\infty}^{+\infty} f_i(x) e^{ikx} dx$ is the Fourier transform of $f_i(x)$, for $i = 1$ or 2. (10 points)

(b) Use the above convolution theorem to solve the integral equation for F(x) where $F(x) = \lambda(x) + \int_{-\infty}^{+\infty} G(x - x') \cdot F(x') dx'$, where $\lambda(x)$ and $G(x - x')$ are given functions. (10 points)

3. (a) Identify the pole and the branch point of $\int_0^\infty \frac{x^{-k}}{1+x} dx$, $0 < k < 1$.

(5 points)

(b) Use the method of contour integration to evaluate $\int_0^\infty \frac{x^{-k}}{1+x} dx$, $0 < k < 1$.

(10 points)

4. Solve $dy/dx + y = x$, $y[0] = 4$.

(10 points)

5. Solve the equation $xy\,dx + (2x^2 + 3y^2 - 20)dy = 0$

(10 points)

6. Solve $x^3 y''' - 2xy' + 4y = \ln x + x^3$

(10 points)

7. Integral transformation: Represent $f(x) = e^{-x}$, $x > 0$ (a) by a cosine integral (b) by a sine integral.

(10 points)

8. Use power series to solve $y'' + xy = 0$.

(10 points)