1. [20 points] An $n \times n$ matrix A is *Hermitian* if $A = A^*$, where A^* is the conjugate transpose of A. It is *normal* if $A^*A = AA^*$ and *unitary* if $AA^* = I_n$, where I_n is the identity matrix. It is *nilpotent* if $A^k = 0$, for some $k \geq 1$.

   Using the above definitions, give a prove for the following statements if they are true or give a counterexample if they are wrong.

   (a) If A is Hermitian, then all eigenvalues of A are real.

   (b) If A is an invertible matrix, then $(A^*)^{-1} = (A^{-1})^*$.

   (c) If A is both Hermitian and unitary, then it is nilpotent.

   (d) If A and B are Hermitian, then AB is Hermitian.

2. [15 points] Given any two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^\top$ in $\mathbb{C}^n$, let $< \cdot, \cdot >$ be the standard inner product on $\mathbb{C}^n$ such that

$$< \mathbf{x}, \mathbf{y} >= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

   (a) If A is Hermitian and λ is an eigenvalue of A with eigenspace

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n | A\mathbf{x} = \lambda \mathbf{x}\},$$

   define

$$E_\lambda^\perp = \{\mathbf{y} \in \mathbb{C}^n | < \mathbf{x}, \mathbf{y} >= 0, \forall \mathbf{x} \in E_\lambda\}.$$

   Prove that $E_\lambda^\perp$ is an invariant subspace of A, that is, $A\mathbf{y} \in E_\lambda^\perp$ for all $\mathbf{y} \in E_\lambda^\perp$.

   (b) Prove that if A is Hermitian, then A is diagonalizable.

   (c) Prove that if A is Hermitian and $A - iI_n$ is invertible, then

$$B = (A - iI_n)^{-1}(A + iI_n)$$

   is unitary. Here $i = \sqrt{-1}$.

3. [15 points] Let A be a 3×2 matrix and B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

   (a) Find the rank of A and the rank of B.

   (b) Prove that there exists a 2×3 matrix C such that $CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. [20 points] Let P_2 be the set of all polynomials with real coefficients of degree less than or equal to two. Define a transformation $T : P_2 \to P_2$ as $T(f(x)) = xf'(x)$.

   (a) Is T a linear transformation?

   (b) Find the matrix of T with respect to the standard basis 1, x, x^2 of P_2.

   (c) Find the dimension of the kernel of T.

   (d) Find all the eigenfunctions of transformation T, i.e., the function $f(x)$ such that $T(f(x)) = \lambda f(x)$ for some scalar λ.

5. [15 points] For each $n \times n$ complex matrix $A = (a_{ij})$, define the notation $\text{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$. Let S be set denoted by

$$S = \{A \in \mathbb{C}^{n\times n} | A^m = I_n, \text{ for some positive integer } m\}.$$

   (a) Prove that $|\text{tr}(A)| \leq n$ for any $A \in S$.

   (b) Find the subset $\{A \in S | \, |\text{tr}A| = n\}$.

6. [15 points] Let $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

   (a) Find the minimal polynomial of A.

   (b) Find A^n, where n is a positive integer.

   (c) Evaluate $\lim_{n\to\infty} \text{tr}(A^n)$.