國立臺灣師範大學 103 學年度碩士班招生考試試題

科目：**線性代數與代數** 適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

1. (6 分) Let $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ be a given $n\times n$ matrix (with coefficients in $\mathbb{R}$). Prove or disprove that there exist invertible matrices $P, Q \in \mathrm{M}_{n\times n}(\mathbb{R})$ such that PAQ is a diagonal matrix.

2. Let V and W be finite dimensional vector spaces over $\mathbb{R}$.

   (1) (6 分) Let $T : V \to W$ be a linear transformation. Show that if $\dim V < \dim W$ then T can not be onto; if $\dim V > \dim W$ then T can not be one to one.

   (2) (10 分) Recall that V and W are called isomorphic if there exists a one-to-one and onto linear transformation from V to W. Show that V and W are isomorphic if and only if $\dim V = \dim W$,

   where $\dim V$ denotes the dimension of the vector space V over $\mathbb{R}$.

3. (8 分) Let $b_1, \ldots, b_4 \in \mathbb{R}$ and consider the following system of linear equations:

$$\begin{aligned} 2x_2 + 4x_3 + 2x_4 + 2x_5 &= b_1 \\ 4x_1 + 4x_2 + 4x_3 + 8x_4 \quad &= b_2 \\ 8x_1 + 2x_2 \quad + 10x_4 + 2x_5 &= b_3 \\ 6x_1 + 3x_2 + 2x_3 + 9x_4 + x_5 &= b_4 \end{aligned}$$

   Let V be the subset of tuple $(b_1, b_2, b_3, b_4) \in \mathbb{R}^4$ such that the above equation is solvable. Show that V is a subspace of $\mathbb{R}^4$ and compute its dimension.

4. (10 分) Let W be the subspace of $\mathbb{R}^5$ defined by the following homogenous equations:

$$\begin{aligned} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 &= 0 \\ 3x_3 - 2x_4 \quad &= 0 \end{aligned}$$

   The orthogonal projection of $\mathbf{v} \in \mathbb{R}^5$ on W is the vector $\mathbf{u} \in W$ such that $\mathbf{v} - \mathbf{u} \in W^\perp$ (the orthogonal complement of W). Show that $\mathbf{u}$ depends uniquely on $\mathbf{v}$. Denote the vector $\mathbf{u}$ by $\mathbf{Proj}_W(\mathbf{v})$. Compute $\mathbf{Proj}_W(\mathbf{v})$ for $\mathbf{v} = (23, -48, 0, 0, -32)$.

5. (10 分) Let $n \geq 2$ be an integer and let T be the linear operator on $\mathbb{R}^n$ such that

$$T(\mathbf{e}_i) = \mathbf{e}_{i+1} \quad \text{for } i = 1, \ldots, n-1 \text{ and } T(\mathbf{e}_n) = \mathbf{e}_1,$$

   where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$. Show that 1 is an eigenvalue of T. Is T diagonalizable ? You need to explain your answers.

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6. (12 分) Let G be a finite group of order n. Determine each of the following statements true or false.

   (1) If $9 \nmid n$, then G has no subgroup of order 6.

   (2) If $3 \mid n$, then G contains an element of order 3.

   (3) If $4 \mid n$, then G contains a subgroup of order 4.

   (4) If $6 \mid n$, then G contains a subgroup of order 6.

   (5) If $4 \mid n$ and G is abelian, then G contains an element of order 4.

   (6) If $6 \mid n$ and G is abelian, then G contains an element of order 6.

7. Let H be a subgroup of a group G.

   (1) (3 分) Give a definition for H being a normal subgroup of G.

   (2) (5 分) Prove that the alternating group A_5 is a normal subgroup of the symmetric group S_5.

8. (6 分) Let G be a group of order 39. Prove that G is not simple.

9. Consider the map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(k) = 2k$ for all $k \in \mathbb{Z}$.

   (1) (4 分) Is ϕ a group homomorphism? Prove your answer.

   (2) (4 分) Is ϕ a ring homomorphism? Prove your answer.

10. (6 分) Prove that the polynomial $x^6 + x^3 + 1 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

11. (7 分) Consider the polynomial $f(x) = x^4 + x^2 + 1$ in $\mathbb{Z}_2[x]$ and let $I = \langle f(x) \rangle$ be the principal ideal generated by $f(x)$ in $\mathbb{Z}_2[x]$. Is I a maximal ideal? Explain your answer.

12. (3 分) Find a maximal ideal in the field $\mathbb{Q}$. No explanation is needed for your answer.

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