國立臺灣師範大學 103 學年度碩士班招生考試試題

科目：高等微積分　　　　適用系所：數學系

注意：1. 本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。
2. 答案必須寫在指定作答區內，否則不予計分。

---

1. (10 分)

   (a) Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < \lambda < f(b)$, then there exists a point $\hat{x} \in (a,b)$ such that $f(\hat{x}) = \lambda$.

   (b) Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $[a,b] \subset f([a,b])$, then there exists a point $u \in [a,b]$ such that $f(u) = u$.

2. (10 分) Prove that every compact metric space is a complete metric space, but the converse is not true.

3. (10 分) Let $\{K_\alpha \mid \alpha \in \Lambda\}$ be a family of closed subsets of a metric space X. If $\bigcap_{\alpha \in \mathscr{B}} K_\alpha \neq \varnothing$ for each finite subset $\mathscr{B} \subset \Lambda$, and there exists an $\alpha_0 \in \Lambda$ such that K_{α_0} is compact, then $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \varnothing$.

4. (10 分)

   (a) Let $\{s_n\}$ be a sequence of positive real numbers. Prove that if $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = s$ then $\lim_{n\to\infty} \sqrt[n]{s_n} = s$.

   (b) Evaluate $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}}$.

5. (10 分)

   (a) Let $\{f_n\}$ be a sequence of real-valued continuous functions on a metric space (X,d). Suppose that $\{f_n\}$ converges uniformly to the function $f : X \to \mathbb{R}$. Prove that f is continuous on X.

   (b) Let

$$g(x) = \sum_{n=0}^{\infty} \frac{\sin nx + \cos nx}{n^2}, \qquad x \in \mathbb{R}.$$

   Is g continuous on $\mathbb{R}$? Explain your reason.

(背面尚有試題)

6. (10 分)

(a) Let $f : [a,b] \to \mathbb{R}$ be continuous and $f(c) > 0$ at some point $c \in [a,b]$. Show that the definite integral $\int_a^b f(x)\,\mathrm{d}x > 0$.

(b) Let $g : [a,b] \to \mathbb{R}$ be a function such that $g(x) = 5566$ except at finitely many points $c_1, c_2, \ldots, c_n$. Show that g is Riemann integrable on $[a,b]$.

7. (10 分) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose that $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$. Define

$$H(x,y) = \int_0^x f(t,0)\,\mathrm{d}t + \int_0^y g(x,t)\,\mathrm{d}t, \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

Compute $\dfrac{\partial H}{\partial x}$ and $\dfrac{\partial H}{\partial y}$.

8. (10 分)

(a) Let $a > 0$. Evaluate the integral: $\displaystyle\int_0^a \int_x^a \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$.

(b) Given $\displaystyle\int_0^1 (1-x)f(x)\,\mathrm{d}x = 4$, find $\displaystyle\int_0^1 \int_0^x f(x-y)\,\mathrm{d}y\,\mathrm{d}x$.

9. (10 分) Let $I = (a,b)$ be an open interval, and a function $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Show that for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $a < x \le c \le y < b$ and $0 < |x-y| < \delta$, we have

$$\left| \frac{f(x) - f(y)}{x - y} - f'(c) \right| < \varepsilon.$$

10. (10 分) For each positive integer $n \in \mathbb{N}$, let r_n be the unique positive real number satisfying $r_n + r_n^3 = n$.

(a) Show that $\{(\sqrt[3]{n} - r_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence.

(b) Find the limit: $\lim\limits_{n\to\infty} (\sqrt[3]{n} - r_n)$.

(試題結束)