國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：線性代數【通訊所碩士班甲組】

題號：4087

1. Given the following matrix: (12%)

$$\begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$$

Determine whether it is Hermitian, unitary, singular, and positive definite. Please explain your reasons to each answer.

2. Consider the following 3×3 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(i) Find the eigenvalue decomposition of $\mathbf{A}$ (8%)

(ii) Find a matrix $\mathbf{L}$ such that $\mathbf{L}\mathbf{L}^T = \mathbf{A}$ (5%)

(iii) Find the singular values of the matrix $\mathbf{L}$ (5%)

3. Consider three vectors :

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(i) Apply the Gram-Schmidt process to $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ to form a set of orthonormal bases. (5%)

(ii) Find the orthogonal projection of a vector $\boldsymbol{b} = [2 \quad -1 \quad 3 \quad 1 \quad 1]^T$ on the space spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$. (5%)

(iii) Find the QR decomposition of (5%)

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(iv) Find a solution of $\boldsymbol{x} = [x_1 \quad x_2 \quad x_3]^T$, such that $\|\mathbf{U}\boldsymbol{x} - \boldsymbol{b}\|^2$ is minimized. (5%)

4. (15%) Let $t: \mathcal{P}_2 \to \mathcal{P}_2$ be

$$a_0 + a_1x + a_2x^2 \to (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find the eigenvalues and the associated eigenvectors of the map t.

5. (10%) Show that if the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent set then so is the set $\{\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$.

6. (15%) Show that matrices of this form are not diagonalizable.

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, c \neq 0$$

7. (10%) **T** is said to be positive definite if $\langle \mathrm{T}(x), x \rangle > 0$ for all $x \neq 0$. Let **T** and **U** be positive operators on an inner product space **V**. Prove
   (i) (5%) $\mathbf{T} + \mathbf{U}$ is positive definite.
   (ii) (5%) If $\mathrm{c} > 0$, then $\mathrm{c}\mathbf{T}$ is positive definite.