1. Find the solution for the following differential equation:

$$y^{iv} + 4y = 0, \quad y(0) = \frac{1}{2}, \quad y'(0) = -\frac{3}{2}, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -\frac{7}{2} \qquad (10\%)$$

2. Find the general solution for the following differential equation:

$$y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x} \qquad (10\%)$$

3. Given the 2nd-order differential equation: $y'' + p(x)y' + q(x)y = r(x)$. Assume $y_1(x)$ and $y_2(x)$ are the homogeneous solutions, prove that the particular solution

$$y_p(x) = -\int \frac{y_2 r}{y_1 y_2' - y_2 y_1'} y_1 + \int \frac{y_1 r}{y_1 y_2' - y_2 y_1'} y_2 \qquad (15\%)$$

4. Find a general solution in terms of J_v and Y_v for $y'' + k^2 x^2 y = 0, \; (y = u\sqrt{x}, \; \frac{1}{2}kx^2 = z)$. Indicate whether you could also use J_{-v} instead of Y_v. $\qquad (10\%)$

5. Find solution by using Laplace transform method

$$y'' + y = 2t \quad y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \; y'(\frac{1}{4}\pi) = 2 - \sqrt{2} \qquad (15\%)$$

6. (a) Find the Fourier coefficients and its corresponding Fourier series of the periodic function $f(x)$. The formula is $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x^2 & \text{if } 0 \le x \le \pi \end{cases}$ and $f(x + 2\pi) = f(x)$

   (b) Find the value of $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ $\qquad$ (20%, each 10%)

7. Solve the following PDE:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = A_0, \quad u(0,t) = 0, \; u(\infty,t) = A_0 \qquad (20\%)$$

$$\text{where} \left(L\left[erfc\left(\frac{a}{2\sqrt{t}} \right) \right] = \frac{e^{-a\sqrt{s}}}{s} \right).$$