1. Let $A=\begin{bmatrix}0&0&2\\1&2&6\\3&7&9\end{bmatrix}$. Is the matrix A non-singular? Justify your answer.

   If exists, find A^{-1} and express A as a product of elementary row matrices. (5%)

2. Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ be a matrix whose elements are non-negative and

   satisfy $a+c=1=b+d$. Also let $P=\begin{bmatrix}b&1\\c&-1\end{bmatrix}$.

   Prove that if $A\neq I_2$ then

   (a) P is non-singular and calculate $P^{-1}AP$, (4%)

   (b) $A^n\to\dfrac{1}{b+c}\begin{bmatrix}b&b\\c&c\end{bmatrix}$ as $n\to\infty$, if $A\neq\begin{bmatrix}0&1\\1&0\end{bmatrix}$. (4%)

3. (a) Prove that if A and B are two matrices with m rows, and $N(A^T)\subset N(B^T)$, then $R(B)\subset R(A)$. (4%)

   (b) Let v be a subspace. Show that $(v^{\perp})^{\perp}=v$. (4%)

4. Consider the vectors

   $\mathbf{u}_1=(0,1,0,1,0)$, $\mathbf{u}_2=(1,0,0,0,0)$, $\mathbf{u}_3=(1,0,1,0,1)$,

   $\mathbf{w}_1=(1,1,0,0,0)$, $\mathbf{w}_2=(1,2,0,1,0)$, and $\mathbf{w}_3=(1,1,1,0,1)$.

   Let U be the subspace of $\mathbf{R}^5$ spanned by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ and W be the subspace spanned by $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

   (i) Select bases for U and W from $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ and $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$, respectively. (1%)

   (ii) What are the dimensions of U and W? (1%)

   (iii) Determine $U\cap W$. (1%)

   (iv) Find a basis of $U\cap W$. (1%)

   (v) Extend the basis from (iv) to bases of U and W in such a way that you will get a basis of $U+W=\text{span}(U\cup W)$ as well. What is the dimension of $U+W$? (1%)

5. In $\mathbf{R}^3$, let g be a line through the origin and E be a plane through the origin such that g is not in E. Determine (geometrically) the eigenvalues and eigenspaces of the following linear maps:
   (i) reflection in the plane E. (1%)
   (ii) reflection in the origin. (1%)
   (iii) parallel projection in the direction of g onto E. (1%)
   (iv) rotation about g through $\frac{\pi}{3}$ followed by rescaling in the direction of g with factor 6. (1%)
   (v) Which of these maps admit a basis of eigenvectors? (1%)

6. Write down matrices $A_i \in \mathbf{R}^{(4,4)}$ in Jordan normal form with the following properties:
   (i) A_1 has eigenvalues 2 and 4, with 2 having algebraic multiplicity 3 and geometric multiplicity 1. (1%)
   (ii) A_2 has the eigenvalue 5 with algebraic multiplicity 4 and geometric multiplicity 3. (1%)
   (iii) A_3 has the eigenvalue 7 with algebraic multiplicity 2 and geometric multiplicity 2 and the eigenvalue -3 with algebraic multiplicity 2 and geometric multiplicity 1. (1%)
   (iv) The matrices A_4 and A_5 both have the eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2 and have no other eigenvalues. Furthermore, A_4 and A_5 are not similar. (2%)

7. Consider the matrix $Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (a) Is this matrix positive definite, negative definite, or indefinite? (1%)
   (b) Is this matrix positive definite, negative definite, or indefinite on the subspace $M = \{x : x_1 + x_2 + x_3 = 0\}$? (3%)

   (c) Consider the quadratic form $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 5x_3^2 + 2\xi x_1 x_2 - 2x_1 x_3 + 4x_2 x_3$.

   Find the values of the parameter ξ for which this quadratic form is positive definite. (4%)

8. Find an orthogonal matrix C such that the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is transformed into a diagonal matrix by $C^{-1}AC = C^{t}AC$. Which property of A guarantees that you can find such a C and the corresponding diagonal matrix ? (6%)

9. Let X and Y be independent random variables with exponential probability density function with parameter, λ. (8%)
   (1) What is the cumulate density function of $Z=X+Y$?
   (2) What is the conditional probability density function $f_{X|X+Y}(x)$?

10. Find the probability that among 10,000 random digits, the digit 7 appears not more than 971 times. (please give your answer in terms of $\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}dx$). (5%)

11. Let X and Y be two random variables with positive variances.
   (1) Let $\hat{X}_L$ be the linear least squares estimator of X based on Y. Show that
   $E\left[(X-\hat{X}_L)Y\right]=0$. (5%)
   (2) Let $\hat{X}$ be the Bayesian estimator, $\hat{X} = E[X \mid Y]$.
   Show that for any function h, $E\left[(X-\hat{X})h(Y)\right]=0$. (5%)
   (3) Is it true that the estimation error $X-E[X \mid Y]$ is independent of Y? (3%)

12. The number of failures of a computer network is assumed to possess Poisson distribution. The mean time to failure of the network is 3 months. What is the probability that the network will not fail within two years? Derive your exact answer from two possible distributions (Hint: define two random variables first). (10%)

13. (1) Assume X is normally distributed with parameters μ and σ, please find the moment-generating function of X. (7%)
   (2) Let X and Y are jointly normal, please find the variance of $Z = X + Y$, and prove that X and Y are uncorrelated then they are independent. (7%)