國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：離散數學【資訊工程學系碩士班】

題號：4081

There are 9 problems in this test. Write down detailed steps for the solution to each problem. Otherwise, no credits for that problem will be given.

1. (10%) Assume that a sequence of numbers is defined by $x_0 = 0$, $x_1 = 1$, and $x_n + 2x_{n-1} = 15x_{n-2}$ for $n > 1$. Find the generating function for the sequence, and then find an explicit expression for x_n.

2. (10%) Solve the following recurrence equation for $T(n)$, $n > 0$.

$$T(n) = \begin{cases} 0, & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + \log n, & \text{if } n > 1. \end{cases}$$

3. (10%) Assume that, for any two people x and y, x is a friend of y if and only if y is a friend of x. Show that, in any group of two or more people, there are always two people with exactly the same number of friends inside the group.

4. (10%) Let G be a simple graph. A *path* of G is a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ such that v_{i-1} and v_i are adjacent for each $i = 1, 2, \ldots, k$. The *length* of the path $v_0, v_1, \ldots, v_k$ is k. The *degree* of a vertex is the number of edges incident to that vertex. Show that if the minimum degree of G is greater than or equal to k, then G has a path whose length is at least k.

5. (15%) A *lattice path* from (x_0, y_0) to (x_n, y_n) in the xy plane is defined as a sequence of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ such that each $x_{i+1} = x_i + 1$, and each $y_{i+1} = y_i \pm 1$, $i = 1, 2, \ldots, n-1$. How many lattices paths are there from $(0, 1)$ to $(10, 3)$? How many of them do not touch or cross the x axis?

6. (10%) Fibonacci numbers are defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Show that f_{3k} is even, for every positive integer k.

7. (10%) Let $S = \{1, 2, \ldots, n\}$.

   (a) Prove that if n is even then any $n/2 + 1$ subset of S contains two numbers whose sum is $n + 1$.

   (b) In general, determine the value k such that any k subset of S contains two numbers whose sum is $n + 1$.

8. (10%) A graph G can be embedded in a plane if it can be drawn in the plane in such a way that no edges intersect. Similarly, a graph G can be embedded in the surface of a sphere if it can be drawn in the surface in such a way that no edges intersect. Show that, for any graph G, G can be embedded in a plane if and only if G can be embedded in the surface of a sphere.

9. (15%) A *theorem* of a propositional calculus is a proposition that is always true. Determine which of the followings are theorems. Justify your answers.

   (a) $(x \Rightarrow x)$

   (b) $\neg(x \Leftrightarrow x)$

   (c) $(((x \Rightarrow y) \wedge (\neg(x) \Rightarrow y)) \Rightarrow y)$

   Note that the symbol $\Rightarrow$ denotes `imply`, $\Leftrightarrow$ denotes `if and only if`, $\wedge$ denotes `and`, $\vee$ denotes `or`, and $\neg$ denotes `not`.