國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：線性代數【電機系碩士班己組】

題號：4060

1. (5%)(單選題) Given $n \times n$ matrices **A**, **B** and **S**. Among the following statements, which are **true**?

   [i] $tr(\mathbf{AB}) \neq tr(\mathbf{BA})$
   [ii] $tr(\mathbf{SAS}^{-1}) = tr(\mathbf{A})$
   [iii] $\mathbf{AB} - \mathbf{BA} \neq \mathbf{I}$
   [iv] $det(\propto \mathbf{A}) = \propto^n det(\mathbf{A})$, $\propto$ is constant
   [v] $det\,(\mathbf{A}^T) \neq det\,(\mathbf{A})$

   (a) i、ii、iii (b) i、ii、v (c) i、iii、iv (d) ii、iii、iv (e) iii、iv、v

2. (5%)(單選題) Let $\mathbf{A}_T$ be the matrix representation of the linear transformation T.
   $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y + 3z \\ -2x + 4y - 6z \\ 3x - 6y + 9z \end{bmatrix}$$

   Let $\mathbf{A}_T$ be the matrix representation of the linear transformation T.
   Which statement is **not** correct?

   (a) The matrix $\mathbf{A}_T$ is linearly dependent.
   (b) The kernel of T is span$\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$.
   (c) The range of T is span$\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right\}$.
   (d) The nullity of T is 2.
   (e) The rank of T is 1.

3. (10%) Find a singular value decomposition of $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$.

4. (15%) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

   (i) Find the characteristic function of **A**. (5%)
   (ii) Compute $\mathbf{A}^5 - 7\mathbf{A}^4 + 13\mathbf{A}^3 - 13\mathbf{A}^2 + 7\mathbf{A} - \mathbf{I}$. (10%)

5. (15%) Let $\{f_1(x), f_2(x), f_3(x)\}$ be a basis for a vector space, where

$$f_1(x) = \begin{cases} 1 & if\ 0 \le x \le 1 \\ 0 & otherwise \end{cases}, f_2(x) = \begin{cases} 1 & if\ 1 \le x \le 2 \\ 0 & otherwise \end{cases},$$
$$f_3(x) = \begin{cases} 1 & if\ 2 \le x \le 3 \\ 0 & otherwise \end{cases}.$$

Define a linear transformation $L[.]$ having the following properties:

$$L[f_1(x)] = \begin{cases} 1 & if\ 0 \le x \le 2 \\ 0 & otherwise \end{cases}, L[f_2(x)] = \begin{cases} 1 & if\ 1 \le x \le 3 \\ 0 & otherwise \end{cases}$$
$$L[f_3(x)] = \begin{cases} 1 & if\ 0 \le x \le 3 \\ 0 & otherwise \end{cases}$$

(i) Find the matrix representation of L with respect to $\{f_1(\mathrm{x}), f_2(\mathrm{x}), f_3(\mathrm{x})\}$. (5%)

(ii) If we use another basis $\{g_1(\mathrm{x}), g_2(\mathrm{x}), g_3(\mathrm{x})\}$, where

$$g_1(x) = \begin{cases} 1 & if\ 0 \le x \le 3 \\ 0 & otherwise \end{cases}, g_2(x) = \begin{cases} 1 & if\ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$
$$g_3(x) = \begin{cases} 1 & if\ 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

Find the matrix representation of L with respect to $\{g_1(x), g_2(x), g_3(x)\}$. (10%)

6. (30%) Consider the following 4×4 matrix **A**

$$\mathbf{A} = \sum_{i=1}^{4} \lambda_i \mathbf{u}_i \mathbf{u}_i^H$$

where λ_i is real and nonzero, and

$$\boldsymbol{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

(i) Prove that **A** is Hermitian (5%)

(ii) Prove that $\boldsymbol{u}_i$ ($i = 1,2,3,4$) are eigenvector of the matrix **A** (5%)

(iii) Find the inverse matrix of the matrix **A** (5%)

(iv) Find the condition that matrix **A** is positive definite (5%)

(v) Find the condition that matrix **A** is unitary (5%)

(vi) Find a matrix **L** such that $\mathbf{L}\mathbf{L}^H = \mathbf{A}$ (5%)

7. (20%) Consider three vectors :

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(i) Apply the Gram-Schmidt process to $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ to form a set of orthonormal bases. (5%)

(ii) Find the orthogonal projection of a vector $\boldsymbol{b} = [2 \quad -1 \quad 3 \quad 1 \quad 1]^T$ on the space spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$. (5%)

(iii) Find the QR decomposition of (5%)

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(iv) Find a solution of $\boldsymbol{x} = [x_1 \quad x_2 \quad x_3]^T$, such that $\|\mathbf{U}\boldsymbol{x} - \boldsymbol{b}\|^2$ is minimized. (5%)