1. Multiple Choice (21%)

Instructions:

- There are 7 questions, each of which is associated with 5 possible responses.
- For each of questions, select **ONE most appropriate** response.
- For each response you provide, **you will be awarded 3 marks if the response is correct and -3 marks if the response is incorrect (答錯一題倒扣三分).**
- You get 0 mark if no response is provided.

(1.1) Consider the ODE $\ddot{x} + \dot{x}^2 + (x^3 - 1) = 0$, where x is a real function.

(A) This is a time-varying ODE.
(B) This ODE is nonlinear.
(C) This ODE has two equilibria.
(D) The equilibria of this ODE are 0 and 1.
(E) None of the above is TRUE.

(1.2) What is the amplitude of the sinusoidal solution of $\ddot{x} + \dot{x} + 2x = 2\sin(t)$?

(A) 2 (B) 1 (C) $\sqrt{2}$ (D) $1/\sqrt{2}$ (E) None of the above.

(1.3) Let $\boldsymbol{L}[\cdot]$ denotes the Laplace transform.

(A) The Laplace transform is a linear operation.
(B) If $\boldsymbol{L}[f(t)] = F(s)$, then $\boldsymbol{L}[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$.
(C) If $\boldsymbol{L}[f(t)] = F(s)$, then $\boldsymbol{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$.
(D) If $\boldsymbol{L}[f(t)] = F(s)$ and $\boldsymbol{L}[g(t)] = G(s)$, then $\boldsymbol{L}[f * g(t)] = F(s)G(s)$, where * denotes the convolution integral.
(E) All of the above statements are TRUE.

(1.4) Define the *del operator* $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

(A) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, where **F** is a vector field with continuous first and second derivatives.
(B) $\nabla \times \nabla F = 0$, where F is a scalar function with continuous first and second derivatives.
(C) $\nabla \cdot (\nabla F \times \nabla G) = 0$, where F and G are scalar functions with continuous first and second derivatives.
(D) All of the above statements are TRUE.
(E) None of the above statements is TRUE.

(1.5) The point on the plane $2x + y - z = 6$ which is closest to the origin is

(A) (2,1,-1) (B) (2,2,0) (C) (3,0,0) (D) (1,1,-2) (E) None of the above

(1.6) For what value of b will the solutions to $\ddot{y} + b\dot{y} + y = 0$ exhibit oscillatory behavior ?

(A) 1 (B) 2 (C) 3 (D) -2 (E) All of the above

(1.7) What is the work done by the vector field $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j}$ around the circle of radius 1, centered at the origin, oriented counter-clockwise ?

(A) π (B) $-\pi$ (C) $\frac{3}{2}\pi$ (D) 2π (E) None of the above

2. (9%) Evaluate the following integral

$$\int_0^1 \int_{x^2}^1 xe^{-y^2}\, dydx$$

3. (15%) Find the solution to the following heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall 0 < x < 1, t > 0$$
$$u(0,t) = 0, \qquad u(1,t) = 1 \qquad \forall t > 0$$
$$u(x,0) = x + \sin(\pi x) \qquad \forall 0 < x < 1$$

4. (12%) Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a set of linearly independent vectors in $\mathbb{R}^n$ and let $S := Span(\mathbf{x}, \mathbf{y})$. Define matrix $A := \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{z}^T$. Obviously, the sets S, its orthogonal complement $S^{\perp}$, and the four sets associated with matrix A, i.e. the two ranges $R(A)$ and $R(A^T)$ and the two null spaces $N(A)$ and $N(A^T)$, are all subspaces of $\mathbb{R}^n$.

This problem has two questions. The first one is a MULTIPLE-choice question, which you don't need to give any derivation. But you **need to give detailed derivations for the second question.** For the multiple-choice question, the total score is evenly divided into each correct statement, and your each correct choice will get the partial score. However, the penalty for each wrong choice is equal to the score of each correct choice. (所以同時選了一個對的答案和一個錯的答案時，淨得分為 0；但是扣分僅扣到該小題 0 分為止。另外、為方便改題，請將選擇題的答案寫在此題做答處即可，不要寫到別處，以免漏改。)

(4.1) What are the possible relationships associated with S and $S^{\perp}$? (6%)

(A) $S^{\perp} \subset N(A^T)$

(B) $N(A^T) \subset S^{\perp}$

(C) $S^{\perp} \subset N(A)$

(D) $S \subset R(A)$

(E) $R(A^T) \subset S$

(4.2) Now let $(\lambda, \mathbf{v})$ be an eigenpair of matrix A^T with $\lambda \neq 0$. Then from the definition of A, it can be shown that $\mathbf{v}$ lies in certain subspace of $\mathbb{R}^n$ and λ is an eigenvalue of another matrix, denoted by $B \in \mathbb{R}^{m \times m}$ with $m = rank(A)$. Please (i) (2%) indicate the subspace of $\mathbb{R}^n$ where the eigenvector $\mathbf{v}$ lies, and (ii) (4%) use vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ to describe the matrix B.

5. (13%) Let P_2 denote the vector space of all polynomials of degree less than 2. Consider the transformation $L: P_2 \to \mathbb{R}^2$ defined by $L(p(x)) := \begin{bmatrix} \int_0^{\alpha} p(x) \\ p(\beta) \end{bmatrix}$ with **undecided parameters** $\alpha > 0$ and $\beta \in \mathbb{R}$. Let A be the matrix representation of transformation L with respect to the ordered bases $E=[1, x]$ and $E'=\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$ for P_2 and $\mathbb{R}^2$, respectively.

以下小題僅需依序寫下答案即可，不需做任何推導。

(5.1) Find the set of $\begin{bmatrix}\alpha\\\beta\end{bmatrix}\in\mathbb{R}^2$ such that matrix A becomes singular. (4%)

(5.2) Let's define an inner product for P_2 by $\langle p(x), q(x)\rangle := \sum_{i=1}^{2} p(x_i)q(x_i)$, for arbitrary $p(x), q(x)\in P_2$, with $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\gamma\\1\end{bmatrix}$ and $\gamma\neq 1$ an **undecided parameter**. Find the orthonormal basis, denoted by $F:=[\mathbf{f}_1, \mathbf{f}_2]$, of P_2 generated from basis E given above to satisfy the subspace equality constraints $Span(\mathbf{f}_1)=Span(1)$ and $Span(\mathbf{f}_1, \mathbf{f}_2)=Span(1, x)$. (5%)

(5.3) Let B denote the matrix representation of transformation L with respect to the ordered bases F computed in (4.2) and $F'=\left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right]$ for P_2 and $\mathbb{R}^2$, respectively. Find the matrix B. (4%)

6. (a)(7%) Let $f(z)$ be a complex function defined by

$$f(z)=\begin{cases}\bar{z}^2/z, & \text{if } z\neq 0\\ 0, & \text{if } z=0\end{cases},$$

where $\bar{z}$ denotes the complex conjugate of the complex variable z. Does the function $f(z)$ satisfy the Cauchy-Riemann equations? Give your reason (no credit will be given if there is no explanation).

(b)(8%) Does the derivative of $f(z)$ at $z=0$, i.e., $f'(0)$, exist? Give your reason (no credit will be given if there is no explanation).

7. (15%) Using the theory of Residues, compute the inverse $f(t)$, $-\infty<t<\infty$, of the Fourier transform

$$F(\omega)=\frac{2a}{a^2+\omega^2},\quad a>0.$$