國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：高等微積分【應數系碩士班丙組】

題號：4050

每題佔20%，總分100%。答題時，每題須寫下題號與詳細步驟。

1. Let $f:\ \mathbf{R}^2 \to \mathbf{R}$ be continuous.
   (a) (10%) Prove or disprove: f maps open sets in $\mathbf{R}^2$ to open sets in $\mathbf{R}$.
   (b) (10%) Prove or disprove: f is differentiable on $\mathbf{R}^2$.

2. (a) (6%) Find the Taylor polynomial $P_n(x)$ of the function $f(x) = \frac{1}{1+x}$ at the point $x_0 = 0$. Give the error term R_n also.
   (b) (6%) Show that for any $|x| < 1$, $\lim\limits_{n\to\infty} R_n(x) = 0$.
   (c) (8%) Hence or otherwise, show that $\tan^{-1}(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ when $|x| \le 1$.
   (You may assume without proof that $\tan^{-1}(x) = \int_0^1 \frac{1}{1+x^2}\,dx$.)

3. Let $f:\ [a,b] \to \mathbf{R}$ be Lipschitz, that is, there is some $K > 0$ such that $|f(x) - f(y)| \le K|x-y|$ for any x, y in $[a,b]$.
   (a) (6%) Show that f is continuous at any point $x_0 \in [a,b]$.
   (b) (8%) If $P = \{a = x_0, x_1 \ldots, x_n = b\}$ is a partition of $[a,b]$, show that

   $$0 \le U(f,P) - L(f,P) \le K(b-a)\|P\|,$$

   where $U(f,P)$ and $L(f,P)$ are the upper sum and lower sum of f with respect to P, and $\|P\| := \max\limits_{1\le k\le n}(x_k - x_{k-1})$ is the norm of P.
   (c) (6%) Hence or otherwise, show that f is integrable on $[a,b]$.

4. Let $f:\ \mathbf{R}^n \to \mathbf{R}$ be differentiable at $\mathrm{a} \in \mathbf{R}^n$, with $f(\mathrm{a}) > 0$.
   (a) (10%) Show that there is some $\delta > 0$ such that $f(\mathrm{a}+\mathrm{h}) > 0$ for any $\|\mathrm{h}\| < \delta$.
   (b) (10%) Show from definition that the function $g(\mathrm{x}) = \frac{1}{f(\mathrm{x})}$ is differentiable at a with $\nabla g(\mathrm{a}) = \frac{-\nabla f(\mathrm{a})}{f(\mathrm{a})^2}$.

5. Let $f:\ \mathbf{R} \to \mathbf{R}$ be continuous such that $\lim\limits_{x\to\infty} f(x) = L_+$, $\lim\limits_{x\to-\infty} f(x) = L_-$, $(L_+, L_- \in \mathbf{R})$.
   (a) (10%) Prove or disprove: f achieves its maximum value.
   (b) (10%) Prove or disprove: If f is also differentiable on $\mathbf{R}$, then $\lim\limits_{x\to\infty} f'(x) = 0$.

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