國立嘉義大學九十七學年度
應用數學系碩士班（甲組）招生考試試題

科目：高等微積分

說明：本考試試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。（1~3 題每題 20 分，4~5 題每題 15 分，第 6 題 10 分，共 100 分）

1. (a) Let $f:\Re\to\Re$ be an additive function, i.e., $f(x+y)=f(x)+f(y)$ for all $x, y\in\Re$. If f is continuous at $x=0$, show that f is continuous everywhere on $\Re$. (10%)

   (b) Let $g:\Re\to\Re$, where $g(x)=\begin{cases}-2x & \text{if } x \text{ is a positive rational number,}\\ 2x & \text{if } x \text{ is a negative rational number,}\\ 0 & \text{therwise.}\end{cases}$

   Determine the continuity of g and justify your answer. (10%)

2. Determine whether the following statements true or false and justify your answers. (Hint: Prove your answer or find a counter example.)

   (a) Let $\{u_i\}$ be a sequence of real numbers. If $\sum_{i=1}^{\infty} u_i$ converges, then $\sum_{i=1}^{\infty}(u_i)^3$ converges, too. (5%)

   (b) Let $[a, b]\times[c, d]\equiv\{(x, y)\in\Re^2 \mid x\in[a, b], y\in[c, d]\}$ and $f:[a,b]\times[c,d]\to\Re$ be a real value function. If $\int_a^b\left(\int_c^d f(x, y)\,dy\right)dx$ and $\int_c^d\left(\int_a^b f(x, y)\,dx\right)dy$ are exist, then $\int_a^b\left(\int_c^d f(x, y)\,dy\right)dx=\int_c^d\left(\int_a^b f(x, y)\,dx\right)dy$. (5%)

   (c) Let A be a compact subset of $\Re^n$. If the function f is continuous on A, then $f:\Re^n\to\Re^m$ is uniformly continuous on A. (5%)

   (d) $f(x)=\sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2}\right)x^3$ is continuous on $\Re$. (5%)

3. Let $f:\Re^2\to\Re$ and $g:\Re^2\to\Re$ real-valued functions, where

   $$f(x, y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y)\neq(0, 0),\\ 0 & \text{otherwise,}\end{cases} \quad\text{and}\quad g(x, y)=\begin{cases}\dfrac{x^4+y^4}{\sqrt{x^2+y^2}} & \text{if } (x, y)\neq(0, 0),\\ 0 & \text{otherwise.}\end{cases}$$

   Determine whether f and g are differentiable at $(0, 0)$ and justify your answer. (20%)

4. Prove the following statements:

   (a) If $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series, then the series $\sum_{n=0}^{\infty} a_n\cos(nx)$ is absolutely and uniformly convergent on the interval $[-\pi, \pi]$. (8%)

   (b) If $\{b_n\}$ is a decreasing sequence of positive numbers and the series $\sum_{n=1}^{\infty} b_n\cos(nx)$ is uniformly convergent on the interval $[-\pi, \pi]$, then $\lim_{n\to\infty} n b_n=0$. (7%)

5. Show that every bounded monotone real value function is integrable. (15%)

6. Find the extreme values of the function $f(x, y, z)=4xy+2xz+2yz$ subject to the constraint $xyz=16$. (10%)