國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：線性代數【應數系碩士班乙組、丙組】

題號：4049

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1. (20 points) Let $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be defined as the reflection of the plane through the line $3x=5y$. Represent T as a matrix when we consider the domain of T equipped with the ordered basis B_1 and the range space of T equipped with the ordered basis B_2, where

$$B_1=\{\begin{pmatrix}3\\-5\end{pmatrix},\begin{pmatrix}5\\3\end{pmatrix}\} \quad \text{and} \quad B_2=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\}.$$

2. (20 points) Express the quadratic form

$$f(x,y,z)=x^2+y^2+z^2-2xy+2xz-2yz$$

as

$$\begin{pmatrix}x & y & z\end{pmatrix}A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

for some suitable symmetric 3×3 matrix A. By diagonalizing A, find the maximum and minimum values of f on the unit sphere in $\mathbb{R}^3$, and find all points on the unit sphere where the extrema are assumed.

3. (20 points) Find an invertible matrix S such that $D=S^{-1}AS$ is a diagonalization of the matrix

$$A=\begin{bmatrix}3/4 & 1/4 & 0\\ 1/4 & 3/4 & 0\\ -1/4 & -1/4 & 1/2\end{bmatrix}.$$

Find $\lim_{n\to\infty}A^n$.

4. (20 points) Find a Jordan canonical form and a Jordan basis for the given matrix

$$A=\begin{bmatrix}3 & 6 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & -1\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}.$$

5. (a) (5 points) Prove that the product of all eigenvalues (repeated according to algebraic multiplicities) of an $n\times n$ matrix A is the value of its determinant $\det(A)$. In particular, A is invertible if and only if no eigenvalue of A is zero. (Hint: Use Remainder Theorem.)

(b) (5 points) Let A be a self-adjoint $n\times n$ matrix and $g(x,y)=x^tAy$ be the quadratic form associated to A. Prove that g is positive-definite if and only if A is positive-definite. (Hint: Use diagonalization.)

(c) (5 points) Let A and B be two positive-definite $n\times n$ matrices. Prove that $A+B$ is also positive-definite. In particular, $A+B$ is invertible.

(d) (5 points) Prove that the positive square root of any positive-definite $n\times n$ matrix C is unique. In other words, if A and B are two positive-definite $n\times n$ matrices such that $A^2=B^2=C$ then $A=B$.