答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Suppose the probability of picking a winning house in a race is 0.2, and X is the number of winning pick out of 10 races.

   (a) (5%) Find $P(X = 3)$.

   (b) (10%) Find the moment generating function of X.

2. Let $U, X_1, X_2, \ldots, X_n$ be independent uniform random variables on $[0,1]$. Denote the ordered random variables X_i by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

   (a) (5%) Find $P(U \leq X_{(n)})$.

   (b) (10%) Find $P(X_{(1)} < U < X_{(n)})$.

3. Suppose that X and Y have the joint density function
$$f(x,y) = \frac{3}{2\pi}\sqrt{1 - x^2 - y^2}, \qquad x^2 + y^2 < 1.$$

   (a) (10%) Find $P(X^2 + Y^2 \leq 1/2)$.

   (b) (10%) Find the marginal density of X.

   (c) (5%) Find the conditional density of X given $Y = y$.

4. Let $X_1, X_2, \ldots$ be independent and identically beta distributed random variables with probability density function $f(x) = 12x(1 - x)^2$, $0 < x < 1$. Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) (5%) Find $E\left[\dfrac{1}{X_1 + 1}\right]$.

   (b) (10%) Find the conditional expectation of Y_n given $X_1 = 3/5$.

   (c) (10%) Find the limiting distribution of Y_n.

5. (10%) Jolin and Jay alternate rolling a pair of fair dice, stopping either when Jolin rolls the sum 5 or when Jay rolls the sum 8. Assuming that Jolin rolls first, find the probability that the final roll is made by Jolin.

6. (10%) Suppose an ordinary deck of 52 cards is shuffled and the cards are then turned over one at a time until the first ace appears. Given that the first ace is the 22th card to appear, find the conditional probability that the card following it is the two of spades.

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