I. (25%) Answer the following questions.

1. **(5%)** Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad and \quad B = \begin{bmatrix} 4a+5d+6g & 4b+5e+6h & 4c+5f+6i \\ 2a+3d & 2b+3e & 2c+3f \\ a & b & c \end{bmatrix}.$$

If the determinant of A is 5, what is the determinant of B?

2. **(10%)** Consider the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 4 & 1 & s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ t \end{bmatrix}$$

If the matrix A is **not** invertible, what is the value of s? Apply this value of s and find the value of t that makes the linear system $A\mathbf{x} = \mathbf{b}$ have a solution.

3. **(10%)** For the linear transformation $T : R^3 \to R^2$ with $T(a_1, a_2, a_3) = (a_1 - 2a_2, 3a_3)$, find a basis for the null space of T, and compute the nullity and rank of T.

II. (25%) Answer the following questions.

1. (15%) Given a matrix

$$A=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 1 & 2 & 0\end{bmatrix}.$$

(a) (10%) Use the Gram-Schmidt process to factor A into a product QR, where $Q=(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ is a matrix with orthonormal column vectors $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$ and R is an upper triangular matrix with positive diagonal entries.

(b) (5%) If $\mathbf{x}=2\mathbf{q}_1+2\mathbf{q}_2+\mathbf{q}_3$ and $\mathbf{y}=5\mathbf{q}_1+\mathbf{q}_3$, determine the vector norm $\|\mathbf{x}\|_2$ and the inner product $<\mathbf{x},\mathbf{y}>$.

2. (10%) Suppose that a 3×3 matrix A has eigenvalues λ=0, -1, and 1 with the corresponding eigenvectors $(0,1,-1)^T$, $(0,1,1)^T$, $(1,-1,1)^T$, respectively.

(a) (5%) Find the matrix A.

(b) (5%) Compute A^k for an arbitrary positive integer k.

III. (25%) Answer the following questions.

1. (10%) Consider a continuous random variable X with a normal (Gaussian) distribution with mean 10 and variance 25.
   (a) Determine the probability P(X > 20). Write your answer in terms of the cumulative distribution function Φ of a standard normal random variable Z, i.e. $\Phi(z) = P(Z \le z)$.
   (b) For a random variable Y defined by Y = 2X + 5, what is the probability density function for the random variable Y? What are the mean and variance for Y?

2. (15%) The joint probability density function for two continuous random variables X and Y is given as follows:

$$f(x,y) = \begin{cases} ce^{-(2x+y)} & 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

where c is a constant.

(a) Determine the constant c so that it satisfies the property of a joint probability density function.
(b) Compute the expectation E(X). Show the detailed steps of your derivation.
(c) Compute the conditional probability P(X > 0.5 | Y < 1). Show the detailed steps of your derivation.

IV. (25%) Answer the following questions.

1. (6%) Let X have a geometric distribution.

   (a) Give the probability density function P(X=x) of X.

   (b) Show that $P(X>k+j \mid X>k) = P(X>j)$, where k and j are any nonnegative integers.

2. (6%) Let Y have a binomial distribution with mean 6 and variance 3.

   (a) Give the probability density function P(Y=y) of Y.

   (b) Find P(Y=2).

3. (13%) Let W have a Poisson distribution with the variance 3.

   (a) Give the probability density function P(W=w) of W.

   (b) Find the moment-generating function M(t) of W.

   (c) Find $P(Y \geqq 2)$.