# 國立清華大學命題紙

97 學年度 統計學研究所 碩士班入學考試

科目 機率論 科目代碼 0102 共 2 頁第 1 頁 *請在【答案卷】內作答

請在答案卷內詳細寫出計算或導証過程

1. (15%) Consider $k$ urns $U_i$, $i$=1, ..., $k$, each of which contain $m$ white balls and $n$ black balls. A ball is drawn at random from urn $U_1$ and is placed in urn $U_2$. Then a ball is drawn at random from urn $U_2$ and is placed in urn $U_3$ etc. Finally, a ball is chosen at random from urn $U_{k-1}$ and is placed in urn $U_k$. A ball is then drawn at random from urn $U_k$. Compute the probability that this last ball is black.

2. (20%) Suppose that $X$ and $Y$ are two jointly distributed random variable with joint probability density function:

$$f(x,y) = \begin{cases} c \cdot xy(1-x), & \text{for } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $c$ is a constant.

   (a) Find the value of $c$.
   (b) Find the marginal probability density function of $X$ and $Y$.
   (c) Find the conditional probability density function of $Y$ given $X$=$x$.
   (d) Are $X$ and $Y$ independent? Explain your answer.
   (e) Find the probability $P(Y < 1/2 | X > 1/2)$.

3. (10%) Consider certain events which in every time interval $[t_1, t_2]$ $(0 < t_1 < t_2)$ occur according to the Poisson distribution $P(\lambda(t_2 - t_1))$. Let $T$ be the random variable denoting the time which lapses between two consecutive such events. Derive and identify the distribution of $T$ by computing the probability that $T$>$t$.

4. (15%) Let $X_i$, $i = 1, \ldots, n$, be independent random variables such that $X_i$ has continuous and strictly increasing cumulative distribution function $F_i$. Set $Y_i = F_i(X_i)$, $i = 1, \ldots, n$. Show that the random variable

$$Z = -2\sum_{i=1}^{n} \log(1 - Y_i)$$

is distributed as $\chi^2_{2n}$.

5. (15%) The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are $N$ floors above the ground floor and if each person is equally likely to get off at any one of these $N$ floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

6. (10%) Let $X$ be a random variable with moment generating function $M(t)$ and set

$$K(t) = \log(M(t))$$

for those $t$'s for which $M(t)$ exists. Furthermore, suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$ are both finite. Then show that

(a) $\left.\frac{d}{dt}K(t)\right|_{t=0} = \mu$,

(b) $\left.\frac{d^2}{dt^2}K(t)\right|_{t=0} = \sigma^2$.

7. (15%) Let $X_i$, $i = 1, \ldots, n$, be independent random variables distributed as Uniform(0, 1) and set

$$Y_n = \max(X_1, \ldots, X_n) \text{ and } Z_n = n(1 - Y_n).$$

Then show that, as $n \to \infty$, one has:

(a) $Y_n$ converges in probability to one,

(b) $Z_n$ converges in distribution to $Z$, where $Z$ has the Exponential distribution with parameter $\lambda = 1$.