國立臺灣師範大學九十七學年度碩士班考試入學招生試題

代數 科試題（數學系用，本試題共 2 頁）

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注意：1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。

1. (10 pts) Let G be a group and let $\mathbb{Z}(G)$ be the center of G. Show that if $G/\mathbb{Z}(G)$ is cyclic, then G is abelian.

2. (10 pts)

   (a) Find the commutator subgroup of the group S_3.

   (b) Show that the Klien four group $\{ e, (12)(34), (13)(24), (14)(23) \}$ is a normal subgroup of A_4.

3. (10 pts) Show that a group of order 72 is not simple.

4. (10 pts)

   (a) Let $\mathbb{R}$ be the group of real numbers under addition and let $\mathbb{R}^+$ be the group of all positive real numbers under multiplication. Show that the two groups $\mathbb{R}, \mathbb{R}^+$ are isomorphic.

   (b) Show that the automorphism group of $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_n^*$, where $\mathbb{Z}_n^* = \{ [i] \in \mathbb{Z}_n \mid (i,n) = 1 \}$ under multiplication modulo n.

5. (10 pts) Let p be a prime number. If $f(x)$ is an irreducible polynomial of degree n in $\mathbb{Z}_p[x]$, show that there is a polynomial $g(x)$ in $\mathbb{Z}_p[x]$ such that $x^{p^n} - x = f(x)g(x)$.

6. (18 pts) Prove or disprove each of the following statements.

   (a) $1 + \sqrt{-19}$ is a prime element in $\mathbb{Z}[\sqrt{-19}]$.

   (b) $\mathbb{Z}[\sqrt{-19}]$ is a unique factorization domain (UFD).

   (c) If R is a principal ideal domain (PID), then $R[x]$ is a PID.

7. (12 pts) Let R be a ring with 1, not necessarily commutative. Recall that an element x in R is said to be nilpotent if $x^n = 0$ for some positive integer n. Moreover, a subset N of R is called nil if every element in N is nilpotent. Suppose that I and J are nil ideals of R.

   (a) If M is an ideal of R such that $I \subseteq M$ and M/I is a nil ideal of R/I, show that M is nil.

   (b) Is it true that $I + J$ is nil? Explain your answer.

8. (20 pts) Let $E \subseteq \mathbb{C}$ be the splitting field of the polynomial $f(x) = x^4 - 2$ over $\mathbb{Q}$. Let $\alpha = \sqrt[4]{2}$, $i = \sqrt{-1}$.

(a) Prove that $x^4 - 2$ is irreducible over $\mathbb{Q}$.

(b) Determine $[E : \mathbb{Q}]$.

(c) Show that there exists an element σ in $G = \mathrm{Gal}(E/\mathbb{Q})$ such that $\sigma(\alpha) = \alpha i$ and $\sigma(i) = -i$.

(d) Suppose that $H = \langle \sigma \rangle$ is the subgroup of G generated by σ and let $K = \{ x \in E \mid h(x) = x \text{ for all } h \in H \}$ be the fixed field of H in E. Find out K.