中原大學 97 學年度碩士班入學考試

4 月 13 日 14:00~15:30 應用數學系數學組　　　　誠實是我們珍視的美德，
我們喜愛「拒絕作弊，堅守正直」的你！

科目：高等微積分(滿分 150 分)　　　　(共 1 頁第 1 頁)

□可使用計算機，惟僅限不具可程式及多重記憶者　　■不可使用計算機

In the following, each problem awards 10 points.

Part I. Prove or disprove the following statements.

1. Let $\{x_n\}$ be a sequence in $\mathbf{R}$ and $|x_{n+1}-x_n|<\frac{1}{n}$, then $\{x_n\}$ is a Cauchy sequence.
2. If $\limsup x_n=1$, then $x_n \le 1$ for n large enough.
3. Let A and B be sets in a metric space and $\overline{A}$ and $\overline{B}$ be the closure of A and B respectively, then $\overline{A\cup B}=\overline{A}\cup\overline{B}$.
4. Let S be a set with the discrete metric, then S is complete.
5. Every bounded and closed set in a metric space is compact.
6. Let $f:\mathbf{R}\to\mathbf{R}$ be continuous and A be open in $\mathbf{R}$, then $f(A)$ is also open in $\mathbf{R}$.
7. Let $f:\mathbf{R}^2\to\mathbf{R}$ be continuous and $A=\{f(x):\|x\|=1\}$, then A is a closed and bounded interval.
8. Let $f_n(x)=x-x^n$, then f_n converges uniformly on $[0,1]$.
9. Let $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$, then f is continuous on $[0,1]$.

Part II. Prove the following statements.

10. Let $f_n(x)=n^3x^n(1-x)$, then f_n doesn't converge uniformly on $[0,1]$.

    (Hint: Consider the integral $\int_0^1 f_n(x)dx$.)
11. Let A be open in $\mathbf{R}^n$ and $x+A=\{x+y:y\in A\}$, then $x+A$ is also open in $\mathbf{R}^n$.
12. Let A and B are connected sets in a metric space and $A\cap B\neq\phi$, then $A\cup B$ is also connected.
13. Let $f:[0,1]\to\mathbf{R}$ be continuous and one-to-one, then f is either increasing or decreasing.
14. Let $A\subset\mathbf{R}$ be uncountable, then A has a limit (accumulation) point. (Hint: Consider the set of rational numbers and assume by the contrary.)
15. Let $A=\{(x,y)\in\mathbf{R}^2:x^2+y^2=1\}$, then A is compact and connected. (Hint: Consider the mapping $f(\theta)\doteq e^{i\theta}$.)