科目：高等微積分
考試時間：100 分鐘

系所：
應用數學系碩士班甲組
本科原始成績：100 分

是否使用計算機：是

1. (15%) (a) State the definition of compact set.
(b) State the Heine-Borel Theorem.
(c) Give an example of a bounded and closed set that is not compact (please, explain why).

2. (15%) If M is a metric space, prove that every compact subset of M is complete.

3. (15%) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = (x + y + z, x - y - 2xz).$$

Show that we can solve for $(x, y) = \varphi(z)$ near $z = 0$.

4. (15%) Prove that there is no continuous function taking $[0, 1]$ onto $(0, 1)$.

5. (15%) Let $f_n : [1, 2] \to \mathbb{R}$ be defined by $f_n(x) = x / (1 + x)^n$.
(a) Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1, 2]$;
(b) Is it uniformly convergent?
(c) Is $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x) \, dx$?

6. (15%) Evaluate the following double integral:

$$(a) \int_0^1 \int_y^1 e^{y/x} dx\, dy, \quad (b) \int_0^1 \int_0^{\sqrt{1-x^2}} \sin \sqrt{x^2 + y^2} dx\, dy.$$

7. (10%) Show that the cubic polynomials
(a) $p(x) = x^3 + ax^2 + bx + c$ has no extreme values iff $a^2 \leq 3b$;
(b) $q(x) = x^3 + ax + b$ has exactly one zero for $a > 0$.