國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：數理統計　　系所：統計學研究所碩士班　　是否使用計算機：否
考試時間：100 分鐘　　本科原始成績：100 分

1. Let $X_1, \ldots, X_n$ be a random sample and $n > 3$. Let $\bar{X}$ and S^2 denote the corresponding sample mean and sample variance respectively.

   (a) Show that
   $$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2. \ (10\ \%)$$

   (b) Assume that the X_i's have a finite fourth moment, and denote $\theta_1 = E(X_i)$, $\theta_j = E(X_i - \theta_1)^j, j = 2, 3, 4$. Then show that
   $$Var(S^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2). \ (12\ \%)$$

2. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf
   $$f_X(x) = \begin{cases} 1/\theta, & \text{if } 0 < x < \theta; \\ 0, & \text{otherwise.} \end{cases}$$
   Let $X_{(1)} < \ldots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables. (10 %)

3. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = \exp(-(x-\mu))$, where $-\infty < \mu < x < \infty$.

   (a) Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic. (10 %)

   (b) Prove that $X_{(1)}$ and sample variance, S^2, are independent. (12 %)

4. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\bar{X}^2 - 1/n$, where $\bar{X}$ is the sample variance of $X_1, \ldots, X_n$. (10 %)

5. Let $X_1, \ldots, X_n$ be a random sample from the pmf $f(x|p) = p(1-p)^{x-1}$, $x = 1, 2, 3, \ldots$, and $0 < p < 1$. Find the MLE of $\sqrt{p(1-p)}$. (10 %)

6. Define S_X^2 and S_Y^2 are the two sample variance based on two independent samples of size n and m from $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$ respectively. Let s_X^2 and s_Y^2 are the observed values of S_X^2 and S_Y^2. Find a $100(1-\alpha)\%$ confidence interval for σ_X^2/σ_Y^2 based on s_X^2 and s_Y^2. (12 %)

7. Let $X \sim Bionomial(2, \theta)$, $0 < \theta < 1$. Consider testing $H_0 : \theta = 1/2$ versus $H_1 : \theta = 3/4$. Find the UMP level $\alpha = 1/4$ test. (14 %)