國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：機率論
考試時間：100 分鐘

系所：
統計學研究所碩士班
本科原始成績：100 分

是否使用計算機：是

1. Suppose we have a population of r distinct objects labeled $1,2,\cdots,r$. Objects are drawn with replacement until exactly $k \le r$ distinct objects have been obtained. Let S_k denote the size of the sample required. Compute ES_k and $VarS_k$. (15%)

2. Consider the problem of matching n objects, and let i and r denote distinct specified positions.
   (a) What is the probability that a match occurs at position i and no match occurs at position r?
   (b) Given that there is no match at position r what is the probability of a match in position i? (10%)

3. Let Y have a distribution function given by

$$F(y)=\begin{cases} 0, & y<0 \\ 1-e^{-y^2}, & y\ge 0. \end{cases}$$

Find a transformation $G(U)$ such that, if U has a uniform distribution on the interval $(0,1)$, $G(U)$ has the same distribution as Y. (10%)

4. Let $X_1,\cdots,X_n$ be iid with pdf

$$f_X(x)=\begin{cases} \dfrac{a}{\theta^a}x^{a-1}, & if \quad 0<x<\theta \\ 0, & otherwise \end{cases}.$$

Let $X_{(1)}<\cdots<X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(2)}$, $X_{(2)}/X_{(3)}$,? $X_{(n-1)}/X_{(n)}$, and $X_{(n)}$ are mutually independent random variables. Find the distribution of each of them. (15%)

5. Let X be a ransom variable having density f given by

$$f(x)=\begin{cases} 1/18, & x=1,3, \\ 16/18, & x=2. \end{cases}$$

Show that there is a value of δ such that $P(|X-EX|\ge\delta)=VarX/\delta^2$, so that in general the bound given by Chebyshev inequality cannot be improved. (5%)

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6. Let $X_1, X_2, \cdots, X_n$ be iid with moment generating function $M_X(t)$, $-h < t < h$, and let $S_n = \sum_{i=1}^{n} X_i$ and $\overline{X}_n = S_n / n$.
   (a) Show that $P(S_n > a) \le e^{-at}[M_X(t)]^n$, for $0 < t < h$, and $P(S_n \le a) \le e^{-at}[M_X(t)]^n$, for $-h < t < 0$.
   (b) Use the facts that $M_X(0) = 1$ and $M'_X(0) = EX$ to show that, if $EX < 0$, then there is a $0 < c < 1$ with $P(S_n > a) \le c^n$. Establish a similar bound for $P(S_n \le a)$. (15%)

7. Let $X_1, X_2, \cdots, X_n$ be independent χ^2 - distributed random variables, each with 1 degree of freedom. Define Y as

$$Y = \sum_{i=1}^{n} X_i .$$

   Therefore, Y has a χ^2 distribution with n degrees of freedom.
   (a) Use the preceding representation of Y as the sum of the Xs to show that $Z = (Y - n)/\sqrt{2n}$ has an asymptotic standard normal distribution.
   (b) A machine in a heavy-equipment factory produces steel rods of length Y, where Y is a normally distributed random variable with mean 6 inches and variance 0.2. The cost C of repairing a rod that is not exactly 6 inches in length is proportional to the square of the error and is given, in dollars, by $C = 4(Y - \mu)^2$. If 50 rods with independent lengths are produced in a given day, approximate the probability that the total cost for repairs for that day exceeds \$48. (Express the answer in terms of Φ) (15%)

8. Suppose that Y_1 and Y_2 are independent exponentially distributed random variables, both with mean β, and define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 / Y_2$.
   (a) Find the joint density function of (U_1, U_2).
   (b) Are U_1 and U_2 independent? Why? (15%)