1. (15%) Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of $\mathbf{A}$, show that their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

2. (30%) Consider line integrals $\int_c \mathbf{F} \cdot d\mathbf{r} = \int_c F_1 dx + F_2 dy + F_3 dz$, where $\mathbf{F} = (F_1, F_2, F_3)$, $\mathbf{r} = (x, y, z)$ are vectors, prove that this line integral is path independent if and only if

   (a) $\mathbf{F} = \mathbf{grad}\ f = \nabla f$

   or (b) $\oint_c \mathbf{F} \cdot d\mathbf{r} = 0$ (integration around closed curves c always gives 0)

   or (c) $\nabla \times \mathbf{F} = \mathbf{0}$ provided the region enclosed by curve c is simply connected.

   Note: The expression " $\cdot$ " and " $\times$ " represents the dot and cross product, respectively.

3. (20%) In an undamped mass-spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system and the model can be written as

   $y'' + \omega_0^2 y = K \sin \omega_0 t$ (1) ,

   where $y(0) = y'(0) = 0$ and K is constant.

   Solve equation (1) with given initial conditions using Laplace transform.

   Hint: Use the convolution integral theorem : $\mathcal{L}^{-1}(F(s)G(s)) = f * g$

4. (15%) Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$ using contour integral

5. (20%) Solve the non-homogeneous diffusion problem

   $u_t - c^2 u_{xx} = e^{-\alpha x}$ $0 < x < L$, where c and α are constant.

   BC's: $u(0,t) = u(L,t) = 0$

   IC: $u(x,0) = f(x)$