1. Please prove the following addition formulas for cosine and sine: (10%)

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \end{cases}$$

(***Hint***: You can use the linear transformation y=Ax with a rotation matrix A to derive them.)

2. The derivative of a continuous function $f(x)$ is defined by $f'(x)=\dfrac{df}{dx}$ $=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}, \forall x$. Please prove the derivatives of sin(x) and cos(x) are $\sin'(x)=\cos(x)$ and $\cos'(x)=-\sin(x)$, respectively. (10%)

3. Please find the curve through the point (2,4) in the xy-plane having the slope $-y/x$ at each point of the curve. (10%)

4. Please find the particular solution of the initial value problem $y'=-2xy$ for an unknown function $y(x)$ with the initial condition $y(0)=1$. (10%)

5. A real square matrix $A=\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$ is given.

(5a) Please determine the eigenvalues and eigenvectors of A. (10%)

(5b)This matrix A is orthogonal. All orthogonal matrices have some special properties, e.g. ---

① orthogonality : please state the definition of an orthogonal matrix and verify that A is an orthogonal matrix. (5%)

（背面仍有題目,請繼續作答）

② <u>invariance of inner product</u>: An orthogonal transformation preserves the value of the inner product of vectors. What is an orthogonal transformation? Please prove it and verify that it holds for this matrix A. (8%)

③ <u>determinant of an orthogonal matrix</u>: please compute the determinant of A and prove that the determinant of an orthogonal matrix has the value +1 or −1. (8%)

6. Let R be a **closed bounded region** in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ everywhere in some domain containing R. Then $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$. This is the famous **Green's theorem in the plane.**

(6a) Please verify it for $F_1 = 6y$, $F_2 = 2x + 2$ and C the circle $x^2 + y^2 = 1$. (10%)

(6b) After derivation, we have $A = \frac{1}{2}\oint_C (xdy - ydx)$. This formula expresses the area of R in terms of a line integral over the boundary. It has various applications, e.g. the theory of certain **planimeters** (instruments for measuring area) is based on it. Please derive this formula $A = \frac{1}{2}\oint_C (xdy - ydx)$. (12%)

(6c) Please prove that an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the area $A = \pi ab$. (7%)