Show all works

1. Let $\{a_n\}$ be a sequence of real numbers. State the definitions of $\lim_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$. Give an exmaple of $\{a_n\}$ for which $\lim a_n$ exists and another for $\lim a_n$ does not exist. What can you say about $\limsup a_n$? Explain. [5%]

2. State the definition of a metric space X and give an example of metric space that is not a Euclidean space R^k. You need to verify that it is a metric space. Let $x \in X$. Show that the set $\{y \in X : d(x,y) < 1\}$ is open in the metric space you give and graph the set. [5%]

3. State the definition of a compact set K of a metric space X. Let $x \in X$. Show that the set $B_2(x) = \{y \in X : d(x,y) < 2\}$ is not compact by using the definition of compactness. [5%]

4. Prove that every open set in R^1 is the union of at most countable collection of disjoint segments, $\cup_{n=1}^{\infty}(a_n, b_n)$. [10%]

5. For two sequences $\{a_n\}$ and $\{b_n\}$, prove that
(a) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, [5%]
(b) if in additional $\{b_n\}$ converges, $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n$. [5%]

6. Let $\sum_{n=0}^{\infty} c_n$ converge. Show that $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely on $-1 < x < 1$. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Show that f is continuous on $(-1,1)$ and $\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} c_n$. [10%]

7. (a) Give an example of a double sequence $\{a_{ij}\}$ such that $\lim_{i\to\infty}\lim_{j\to\infty} a_{ij} \neq \lim_{j\to\infty}\lim_{i\to\infty} a_{ij}$. Under what conditions for $\{a_{ij}\}$ will the equality hold in the formula? [10%]
(b) Do the same for $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$. [10%]

8. Let f be a continuous mapping of a metric space X into a metric space Y. Prove that $f^{-1}(V)$ is open in X for every open set V in Y. Is the converse true? Prove it or give a counterexample. [10%]

9. If $f(t) = t + 2t^2 \sin\frac{1}{t}$ for $t \neq 0$, and $f(0) = 0$. Find $f'(0)$ and prove that f' is bounded on $(-1,1)$. Does f have an inverse function in some neighborhood of 0? Give an explanation. (Hint: Use the graph of f.) [10%]

10. Let series $\sum_{n=0}^{\infty} a_n$ converge and $a_n > 0$.
(a) Describe a way to get a rearrangement of $\sum_{n=0}^{\infty} a_n$, say $\sum_{n=0}^{\infty} a'_n$ such that $\{a'_n\}$ is a decreasing sequence. [5%]
(b) Show that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a'_n$. (Give a direct proof.) [10%]