[10%] 1. Let V be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

[10%] 2. Let V be a finite dimensional vector space and let W be a subspace. Show that dimension of the quotient space V/W is $\dim V - \dim W$.

[10%] 3. Let $A = (a_{ij})$ be a strictly upper triangular $n \times n$ matrix with real entries, i.e. $a_{ij} = 0$ if $i \geq j$. Let I be the $n \times n$ identity matrix. Show that $I - A$ is invertible and express the inverse of $I - A$ as a function of A.

[10%] 4. Let A be square matrix over $\mathbb{C}$. Prove that the eigenvalues of A are all real if $\bar{A}^t = A$ where $\bar{A}^t$ means the conjugate transpose of A.

[15%] 5. Let A and B be complex $n \times n$ matrices such that $AB = BA$. Show that there is a vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = \mu\mathbf{v}$ for some $\lambda, \mu \in \mathbb{C}$. (That is, $\mathbf{v}$ is a common eigenvector of A and B.)

[15%] 6. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I)$ is linearly independent in W whenever I is a linearly independent set in V. Show that T is independence preserving if and only if T is one-to-one.

[15%] 7. Let $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}$. Determine conditions on a, b, c, and d so that there is only one Jordan block for each eigenvalue of A in the Jordan canonical form of A.

[15%] 8. Let $\{v_1, v_2, \ldots, v_k\}$ be a linearly independent set of vectors in the real inner product space V. Show that there exists a unique set $\{u_1, u_2, \ldots, u_k\}$ of vectors with the property that $(u_i, v_i) > 0$ for all i, and $\{u_1, u_2, \ldots, u_i\}$ is an orthonormal basis for $\text{Span}\{v_1, v_2, \ldots, v_i\}$ for every i.