一、 Data Structures (50%)

1. (20%) An "array" of size N supports the following operations:

(i) INITIAL( ) - initializes all the array elements to be zero.

(ii) WRITE(k, m) - write value m to position k.

(iii) READ(k) - read the value at location k.

(iv) MULTIPLYALL(n) - multiply all elements by value n. Note that this operation can be performed many times with an accumulative effect. The function should return ERROR if n equals to 0.

Here we assume that INITIAL is the first operation to be called, and it is called only once.

Provide an implementation of such "array" that further satisfy the condition (a) and (b), respectively.

(a) All operations except INITIAL must take $O(1)$ time in the worst case. (10%)

(b) Add another operation ZEROALL() - zeroes all the values in the "array" in $O(1)$ time. (10 %)

2. (10%) True or False.

(a) In an undirected graph G, if there is a path between two vertices x and y then in the DFS tree of G, either x is a descendant of y or y is a descendant of x. (2%)

(b) In a directed graph G, if vertex x has both incoming and outgoing edges, its tree in the DFS forest contains more than one vertex. (2%)

(c) A d-ary heap is like a binary heap, but non-leaf nodes have d children instead of 2 children. The running time of the efficient implementation of Extract-Max in a d-ary max-heap with n elements is $\Theta(\log_d n)$. (3%)

(d) A Hamiltonian Path in graph G passes through each node $v \in V$ exactly once. Given a directed acyclic graph $G=(V, E)$, its Hamiltonian path $v_1, v_2, \ldots, v_n$ must be a topological ordering of G. (3%)

3. (20%) Suppose that we are given a key k to search for in a hash table with positions $0, 1, \ldots, n-1$, and suppose that we have a hash function h mapping the key space into the set $\{0, \ldots, n-1\}$. The searching scheme is as follows:

(Step 1) Compute the value $i \leftarrow h(k)$, and set $j \leftarrow 0$.

(Step 2) Probe in position i for the desired key k. If you find it, or if this position is empty, terminate the searching.

(Step 3) Set $j \leftarrow (j+1) \bmod n$, and $i \leftarrow (i+j) \bmod n$, and return to Step 2.

Assume that n is a power of 2.

(a) Show that this scheme is an instance of the general "quadratic probing" scheme. (10%)

(b) Let $h'(k, i)$ denote the i'th probing of the above scheme. Show that $\langle h'(k, 0), \ldots, h'(k, n-1)\rangle$ is a permutation of $\{0, \ldots, n-1\}$, i.e. each number in $\{0, \ldots, n-1\}$ appears in this sequence once. (10%)

（背面仍有題目,請繼續作答）

二、Algorithms (50%)

4. (10%) Solving the recurrence $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log_2 n$ using big-O notation as tight as possible.
5. (5%) Consider the following two problems in which we are given a directed graph $G=(V, E)$ and vertices $u, v \in V$.
   Unweighted shortest path problem: Find a path from u to v consisting of the fewest edges.
   Unweighted longest simple path problem: Find a path from u to v consisting of the most edges.
   (a) (2%) Determine which problem can be solved using the dynamic-programming in polynomial time.
   (b) (3%) Determine which problem cannot be solved using the dynamic-programming in polynomial time.
6. (20%) Consider the problem of finding the 5-vector $x=(x_i)$ that satisfies

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix}$$

   . Determine how many solutions to this problem.
7. (10%) Show the lower bound for any comparison sort algorithm.
8. (5%) How many strongly connected components in a path with n-vertices?