In the following, $\mathbb{R}$ denotes the set of all real numbers, n is any positive integer and $\mathbb{R}^n$ is the Euclidean space containing all n-dimensional real column vectors. For $1 \le i \le n$, let e_i be a column vector whose transpose is equal to

$$(\overbrace{0,...,0}^{i-1},1,\overbrace{0,...,0}^{n-i}).$$

The set $\{e_1,..,e_n\}$ is known as the canonical basis for $\mathbb{R}^n$.

1. (12%) Which descriptions are correct and why? The solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form a plane, line, point, subspace, null space of A, column space of A.

2. Let F be a vector space of dimension n. Let T be a linear transformation on F. We call the matrix $B = (b_{i,j})_{1 \le i,j \le n}$ the matrix of T in the basis $\{v_1, ..., v_n\}$ if

$$Tv_j = \sum_{k=1}^{n} b_{k,j} v_k \quad \forall j = 1, 2, ..., n.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 3c - b \\ 3b - a \\ a \end{bmatrix}.$$

(a) (6%) Verify that T is a linear transformation.

(b) (7%) Find the matrix of T in the canonical basis of $\mathbb{R}^3$.

3. Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{R}^n$ and $\{w_1, ..., w_n\}$ be another basis of $\mathbb{R}^n$. Let C be the matrix whose columns are $w_1, ..., w_n$. The matrix C is then called the matrix of the change of basis from $\{e_1, ..., e_n\}$ to $\{w_1, ..., w_n\}$.

(a) (7%) Prove that the matrix C is invertible.

(b) (7%) Let T be a linear transformation of $\mathbb{R}^n$ and B be the matrix of T in the canonical basis. Regard B as a linear transformation of $\mathbb{R}^n$ under the matrix multiplication. Then the matrix A of T in another basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ satisfies $A = D^{-1}BD$, where D is the matrix changing the basis from $\{e_1, ..., e_n\}$ to $\{v_1, ..., v_n\}$.

4. (a) (7%) Write the ellipsoid $\frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{16} = 1$ in the form of $x^tAx = 1$, where $x^t = (x_1, x_2, x_3)$ is the transpose of x.

(b) (7%) State any three equivalent definitions of a real symmetric matrix A to be positive definite.

(c) (7%) Let $x \in \mathbb{R}^3$ and B be a 3×3 matrix with real entries. Give conditions on B so that $x^tBx = 1$ represents as an ellipsoid in $\mathbb{R}^3$. Verify your claim.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and A be the $n \times n$ matrix of T in the canonical basis of $\mathbb{R}^n$. Assume that T is a projection map, that is, $T^2 = T$.

   (a) (5%) Prove that A has 1 as an eigenvalue with multiplicity at least $\dim R(T)$, where $\dim V$ is the dimension of V and $R(T)$ is the range of T.

   (b) (5%) Show that the eigenvalue of A is either 0 or 1.

   (c) (5%) Prove that A is similar to a diagonal matrix whose entries are either 0 or 1. (Two matrices P, Q are said to be similar if there exists an invertible matrix X such that $Q = XPX^{-1}$.)

   (d) (5%) Show that the trace of A ($\mathrm{tr}A$) satisfies $\mathrm{tr}A = \dim R(T)$.

   (e) (5%) Prove that the similarity in (a) is orthogonal (that is, X is an orthogonal matrix) if and only if A is symmetric.

6. For any $n \times n$ matrix K, we let $K_{i,j}$ denote the (i, j)-th entry of K for $1 \le i, j \le n$. In this setting, K is called a stochastic matrix if

$$0 \le K_{i,j} \le 1, \quad \sum_{j=1}^{n} K_{i,j} = 1 \quad \forall 1 \le i, j \le n.$$

   Assume in the following that K is a stochastic matrix.

   (a) (5%) Let $\rho(K)$ denote the spectrum of K, that is, the set of all eigenvalues of K. Prove that $|\lambda| \le 1$ for all $\lambda \in \rho(K)$ and also $1 \in \rho(K)$.

   (b) (5%) Show that if v is an n-dimensional row vector such that $vK = v$, then $|v|K = |v|$, where $|v| = (|v_1|, ..., |v_n|)$ as $v = (v_1, ..., v_n)$.

   (c) (5%) K is said to be irreducible if, for any $1 \le i, j \le n$, there exists a positive integer $l = l_{i,j}$ such that the (i, j)-th entry of K^l, denoted by $(K^l)_{i,j}$, is positive, where K^l is the matrix obtained by multiplying K itself for l times. Prove that if K is irreducible, then 1 is a simple eigenvalue (that is, its multiplicity is one).