參考用

科目：工程數學B(5003)　校系所組：中大通訊工程學系甲組、乙組
清大電機工程學系乙組、丙組、丁組
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1. Let $X_1, X_2, \cdots\cdots$ denote a sequence of independent, identically distributed random variables with exponential probability density function (pdf)

$$f_{X_i}(x) = \begin{cases} e^{-x} & x \ge 0, \\ 0 & otherwise. \end{cases}$$

(a) (5%) Let n denote a constant, find the pdf of the derived random variable $Y = \sum_{i=1}^{n} X_i$.

(b) (5%) Let N denote a geometric (1/5) random variable with probability mass function (pmf)

$$P_N(n) = \begin{cases} \frac{1}{5}\left(1-\frac{1}{5}\right)^{n-1} & , \quad n = 1, 2, \cdots \\ 0 & , \quad otherwise. \end{cases}$$

What is the moment-generating function (MGF) of $Z = X_1 + X_2 + \cdots\cdots + X_N$?

(c) (5%) Find the pdf of Z.

$\langle Remark \rangle$ The MGF of a random variable X is defined as

$$\phi_X(s) = E\{e^{sX}\}$$
$$= \begin{cases} \int_{-\infty}^{\infty} e^{sx} f_X(x)dx & X \text{ is a continuous random variable} \\ \sum_{x_i \in \Omega} e^{sx_i} P_X(x_i) & X \text{ is a discrete random variable} \end{cases}$$

2. Let **V** be a vector space of continuous functions defined on the interval $[0, 2\pi]$ and $\beta = \{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$ be a basis for **V**.

(a) (6%) Is the set $\mathbf{W} = \{f(t) \in \mathbf{V} : \int_0^{2\pi} f(t)dt = 0\}$ a subspace of **V**? Justify your answer.

(b) (5%) Define T: **V**→**V** by $\forall f(t) = \sum_{i=1}^{n} a_i\phi_i(t) \in \mathbf{V}$, $T(f(t)) = \sum_{i=1}^{n} a_{i-1}\phi_i(t)$ where $a_0 = 1$. Prove that T is a linear transformation.

(c) (4%) Find bases for both the null space of T and the range of T.

3. (7%) Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A=[\mathrm{T}]_\beta$, where $[\mathrm{T}]_\beta$ denotes the matrix representation of linear operator T in the ordered basis β. That is, $f(t) = \det(A-tI_n)$, where det(.) is the determinant of the indicated matrix, and I_n is the n-by-n identity matrix. Prove that this definition of characteristic polynomial of a linear operator is independent of the choice of ordered basis β. That is, $\det([\mathrm{T}]_\beta - tI_n) = \det([\mathrm{T}]_\gamma - tI_n)$ for any ordered bases β and γ of V.

注意：背面有試題

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4. (5%) The set of all polynomials with real coefficients is a vector space denoted by P(R). Let n be a nonnegative integer, and let $P_n(R)$ consist of all polynomials in P(R) having degree less than or equal to n. Let V= P(R) with inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$, and consider the subspace $P_2(R)$ with the ordered basis $\beta=\{x^2, x, 1\}$. Use the Gram-Schmidt process to replace β by an **orthonormal** basis $\{v_1, v_2, v_3\}$ for $P_2(R)$ in the order of $x^2 \rightarrow x \rightarrow 1$.

5. (8%) Suppose that T is a linear operator on a finite-dimensional inner product space V over the field of real number with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Assume that T is self-adjoint. For each i ($1 \le i \le k$), let W_i be the eigenspace of T corresponding to the eigenvalue λ_i, and let T_i be the orthogonal projection of V on W_i. Prove that $T = \lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k$.

6. (10 %) Suppose that X is a Poisson random variable with $P(X=2) = P(X=3)$. Find $P(X=5)$.

7. (10%) Let $X \sim N(0, 1)$ and $-\infty < a < \infty$. Find $E[e^{aX}]$.

8. Consider the following system of three linear equations in three unknowns:

$$\begin{cases} x_1 + x_2 + ax_3 = 1 \\ x_1 + ax_2 + x_3 = 3 \\ ax_1 + x_2 + x_3 = 2a \end{cases},$$

where $a \in R$.

(a) (4%) Find condition on a such that the system has a unique solution.

(b) (8%) Find condition on a such that the system has no solution. Find also condition on a such that the system has many solutions.

(c) (3%) Under the condition obtained in (a), use Cramer's rule to solve the system (no credit without using Cramer's rule).

9. A silicon wafer contains n CPU processor chips. Assume that a single CPU processor chip has failure probability p.

(d) (5%) What is the failure probability of a single silicon wafer?

(e) (5%) What is the probability of at most two failure chips in a single silicon wafer?

10. (5%) Suppose that three numbers are selected one by one, at random and without replacement from the set of numbers $\{1,2,3,\ldots,n\}$. What is the probability that the third number falls between the first two if the first number is smaller than the second?