國　立　清　華　大　學　命　題　紙

97 學年度 經濟學 系（所） 組碩士班入學考試

科目 微積分與統計 科目代碼 4603 共 2 頁第 1 頁 *請在【答案卷卡】內作答

微積分部分(共五十分)：

1. **[12 points.]** In each of the following statements, determine that it is true or false. Explain your answer. You will not get any point if you just answer "true" or "false."

   (a) If a sequence has an infimum, then the sequence is convergent. (3 points)

   (b) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{n^2}, \ldots\}$. Then $A \cup \{0\}$ is a compact set in $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. (3 points)

   (c) Let $f(x) = \sqrt{x^2 + 2x} - x$. Then $\lim_{x\to\infty} f(x)$ is defined. (3 points)

   (d) $f(x) = x^5 - 6x^2 + 4$ has a root in the interval $x \in [0, 1]$. (3 points)

2. **[10 points.]** Find the maximum values of the following functions:

   (a) $f(x, y) = x^4 - x^2y^2 + y$. (5 points)

   (b) $f(x, y) = e^{-(x^2+y^2)}(x^2 + 2y^2)$. (5 points)

3. **[10 points.]**

   (a) Determine whether or not $\int_0^\infty e^{-x^2} dx$ converges. (5 points)

   (b) Determine whether or not $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+2}$ converges. (5 points)

4. **[10 points.]** For the following maximization problem:

$$\begin{aligned} \max \quad & \ln x + \ln y \\ \text{s.t.} \quad & x^2 + y^2 \leq 1, \end{aligned}$$

   find the solution, and verify that it is indeed the maximum.

5. **[8 points.]** The following differential equation:

$$\dot{y}(t) = 2t - y(t)$$

   satisfies $y(1) = 2$.

   (a) Find $y(t)$. (5 points)

   (b) Does there exist a steady state? If so, find it. (3 points)

二、統計（共五十分）

[**Instructions**: Please do all <u>**FIVE**</u> questions and show all your work.]

1. **[10 points]** Let X stand for the rate of return on a security (say, IBM) and Y the rate of return on another security (say, General Motors). Let $\mu_X = \mu_Y = 0.5$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 9$ and $\text{corr}(X, Y) = -0.8$.

   (a) Find $\text{var}[0.5X + 0.5Y]$.

   (b) Is it better to invest equally in the two securities (i.e., diversify) than in either security exclusively? Explain in detail why or why not. [**Hint**: Investors consider both expected rate of return and risk.]

2. **[10 points]** Let X_i be a random sample of size n with mean μ_X and variance σ_X^2. Consider $\bar{X} = \sum_{i=1}^n X_i/n$ and $\hat{\sigma}_X^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n$.

   (a) Is $\hat{\sigma}_X^2$ an unbiased estimator of σ_X^2?

   (b) What is the sampling distribution of $\bar{X}$ if the population is normally distributed?

3. **[10 points]** The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. If there are five children in the family and it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

4. **[10 points]** Let $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. From a sample of size $n = 100$, you obtained $\sum_{i=1}^n Y_i = 590$ and $\sum_{i=1}^n (Y_i - \bar{Y})^2 = 420$. Test $H_o : \mu_Y = 6$ against $\mu_Y \neq 6$ with the 5% significance level.

5. **[10 points]** Suppose that a light bulb manufacturing plant produces bulbs with a mean life of 2000 hours and a standard deviation of 200 hours. An inventor claims to have developed an improved process that produces bulbs with a longer mean life and the same standard deviation. The plant manager randomly selects 100 bulbs produced by the process. She says that she will believe the inventor's claim if the sample mean life of the bulbs is greater than 2100 hours, otherwise she will conclude that the new process is no better than the old process. Let μ denote the mean of the new process. Consider the null and alternative hypothesis $H_o : \mu = 2000$ vs. $H_1 : \mu > 2000$.

   (a) What is the size of the plant manager's testing procedure?

   (b) Suppose that the new process is in fact better and has a mean bulb life of 2150 hours. What is the power of the plant manager's testing procedure?