

科目：高等微積分(1001)

校系所組：中大數學系甲組 清大數學系純粹數學組
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參考用

Advanced Calculus

本試卷共八題

1. (14 points) Let $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
- (a) (9 points) Show that $E(x)$ is differentiable on \mathbb{R} .
- (b) (5 points) Prove that the function $y = E(x)$ satisfies the initial value problem
- $$y' - y = 0, \quad y(0) = 1.$$

2. (12 points) Prove that the function $\frac{\sin x}{x}$ is improperly integrable on $[1, \infty)$.
3. (14 points) Prove that

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous, has first-order partial derivatives everywhere on \mathbb{R}^2 , but f is not differentiable at $(0, 0)$.

4. (12 points) Let $f(x, y) = (e^x + e^y, e^x - e^y)$. Show that f is locally invertible around every point of \mathbb{R}^2 . Does f have a global inverse on \mathbb{R}^2 itself? Give your reason.
5. (12 points) Let

$$F(T) = \frac{1}{2T} \int_{-T}^T e^{-x^2} dx, \quad T > 0.$$

Show that $F(T)$ is a strictly decreasing function in T .

6. (12 points) Find the limit

$$\lim_{n \rightarrow \infty} \frac{\pi}{n^4} \sum_{1 \leq k, l \leq n} k^2 \cos\left(\frac{l}{n}\pi\right).$$

Explain each step of your computation.

7. (12 points) Let A be a $n \times n$ real matrix and $\det A < 0$. Show that there is a positive number δ such that

$$\|Ax\| \geq \delta \|x\|, \quad \text{for all } x \in \mathbb{R}^n.$$

8. (12 points) Evaluate the line integral

$$I = \oint_C \frac{(2x + y)dx + (2y - x)dy}{x^2 + y^2},$$

where C is the oriented counterclockwise circle $\{(x, y) : x^2 + y^2 = 1\}$.